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Mode conversion by double Čerenkov processes in a plasma

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Abstract. It is shown that the conversion of an electromagnetic wave (ie, a transverse photon) to an electrostatic wave (ie, a longitudinal plasmon) and vice versa can occur in a uniform, homogeneous and isotropic plasma by Čerenkov absorption of one kind (ie, the photon or the plasmon) and the subsequent emission of the other kind (ie, the plasmon or the photon, respectively). Furthermore, it is shown that there also exists an additional 'stimulated mode conversion process'. These mode conversions by double Čerenkov processes are of second order in the Heitler–Ma formulation of the quantum time-dependent perturbation theory. Explicit expressions for the cross sections appropriate to these mode conversion processes are also presented.

1. Introduction

It is well known that the electrons in a plasma can emit and absorb both transverse photons and longitudinal plasmons by the familiar Čerenkov processes (Pines and Schrieffer 1962, Tsytovich 1970, Harris 1972, Arunasalam 1968, 1973). These Čerenkov emissions and absorptions of photons and plasmons are a consequence of the interaction Hamiltonian which is proportional to the scalar product of the particle momentum and the vector potentials of the transverse electromagnetic waves and the longitudinal plasma waves. According to the Heitler–Ma formulation of quantum time-dependent perturbation theory (Heitler 1954) these one-photon or one-plasmon emission and absorption processes are first-order processes, that is these processes occur in the familiar 'Fermi golden-rule' approximation of the time-dependent perturbation theory. These first-order processes are direct transitions from one quantum state to another and do not involve a passage through any intermediate virtual states.

In this paper we wish to show that, in the second order in the Heitler–Ma formulation of the time-dependent perturbation theory, mode conversion processes (ie, the conversion of a transverse electromagnetic wave to a longitudinal plasma wave and vice versa) can occur in a uniform, homogeneous and isotropic plasma by Čerenkov absorption of a quantum of one kind (either the photon or the plasmon) and the subsequent emission of a quantum of the other kind (either the plasmon or the photon, respectively). These mode conversions by double Čerenkov processes occur by passage through an intermediate virtual state. Both energy and momentum have to be conserved between the initial and final states of the transition. However, in intermediate transitions to and from virtual states, momentum will be conserved but energy need not be conserved. This is due to the fact that the electron spends only a very short time in the virtual state and by the Heisenberg uncertainty principle it is meaningless to speak of energy conservation for transitions to and from virtual states. Since in the mode conversion process under study a quantum of one kind (either the photon or the plasmon) is absorbed and subsequently a quantum of the other kind (either the plasmon or the photon, respectively) is emitted, it is clear that the final emission process can be either a spontaneous emission or a stimulated emission. Thus we will show that in a plasma there exists not only a 'spontaneous mode conversion' but also a 'stimulated mode conversion'.

In the literature other authors have considered the mode conversion processes in inhomogeneous plasmas. For example, Stix (1965) has considered mode conversion due to density gradients in a plasma, Tidman (1960) has considered mode conversion due to either a density gradient or a temperature gradient in a plasma, Kritz and Mintzer (1960) have considered mode conversion at a density discontinuity in a plasma. However, mode conversion by double Čerenkov processes to be discussed in this paper can occur in a uniform, homogeneous and isotropic plasma. We shall show that for most cases of practical interest the cross sections for these mode conversions by double Čerenkov processes in a plasma are very much larger than the Thompson cross section. Thus these mode conversions by double Čerenkov processes may have some significance in laser fusion experiments and in the understanding of radio wave emission from extraterrestrial plasmas (such as the observed intense radio wave emission from the bright areas of the sun).

2. Review of the basic concepts

The Hamiltonian $\mathscr{H}$ of a plasma (of electrons and ions) and the radiation field (of transverse photons and longitudinal plasmons) may be written

$$\mathscr{H} = H + H_{\text{int}}, \tag{1}$$

where

$$H|n\rangle = E_n|n\rangle \tag{2}$$

and H_{int} is the interaction Hamiltonian that is responsible for the emission, absorption and scattering of photons and plasmons. In general H_{int} will be a sum of many independent interaction terms. One half of these terms represents the interaction of transverse photons and longitudinal plasmons with the electrons. The other half represents the interaction of the photons and plasmons with the ions. For the sake of simplicity we shall neglect the presence of the ions and consider the plasma as a gas of electrons. Then H_{int} will be a sum of four interaction terms. Two of these terms represent the interaction of the photons with the electrons, and the other two terms represent the interaction of the plasmons with the electrons. Of the two interaction terms that represent the interaction of the photons with the electrons, one of these represents the interaction appropriate to Čerenkov emission and absorption of a single photon (ie, the interaction that is proportional to the scalar product between the electron momentum and the vector potential of the photon field), and the other represents the interaction appropriate to two-photon processes such as the scattering (ie, the interaction that is proportional to the square of the vector potential of the photon field). Similarly, of the two terms that represent the interaction of plasmons with electrons, one of these represents the interaction appropriate to Čerenkov emission and absorption of a single plasmon, while the other represents the interaction appropriate to two-plasmon processes. In this paper we are interested only in the two independent interaction terms that are responsible for Čerenkov emission and absorption of a single photon or a single plasmon. That is, we will write

$$H_{\text{int}} = H_0 + H_1, \tag{3}$$

where H_0 represents the electron–photon interaction that is responsible for Čerenkov emission and absorption of a single photon, and H_1 represents the electron–plasmon interaction that is responsible for Čerenkov emission and absorption of a single plasmon. One can show (Tsytovich 1970) that the non-zero matrix elements of H_0 and H_1 are given by

$$\langle N_0+1, \boldsymbol{v}'|H_0|N_0, \boldsymbol{v}\rangle = (N_0+1)^{1/2} M_0(\boldsymbol{\epsilon}_0 \cdot \boldsymbol{v})\delta_{\boldsymbol{v}',\boldsymbol{v}-\hbar\boldsymbol{k}_0/\mu} \tag{4}$$

$$\langle N_0-1, \boldsymbol{v}'|H_0|N_0, \boldsymbol{v}\rangle = N_0^{1/2} M_0(\boldsymbol{\epsilon}_0 \cdot \boldsymbol{v})\delta_{\boldsymbol{v}',\boldsymbol{v}+\hbar\boldsymbol{k}_0/\mu} \tag{5}$$

$$\langle N_1+1, \boldsymbol{v}'|H_1|N_1, \boldsymbol{v}\rangle = (N_1+1)^{1/2} M_1(\omega_1/k_1)\delta_{\boldsymbol{v}',\boldsymbol{v}-\hbar\boldsymbol{k}_1/\mu} \tag{6}$$

and

$$\langle N_1-1, \boldsymbol{v}'|H_1|N_1, \boldsymbol{v}\rangle = N_1^{1/2} M_1(\omega_1/k_1)\delta_{\boldsymbol{v}',\boldsymbol{v}+\hbar\boldsymbol{k}_1/\mu} \tag{7}$$

where

$$M_0 = \left(\frac{4\pi\hbar q^2}{L^3 \partial(\omega_0^2 D_0)/\partial\omega_0}\right)^{1/2}; \tag{8}$$

$$M_1 = \left(\frac{4\pi\hbar q^2}{L^3 \partial(\omega_1^2 D_1)/\partial\omega_1}\right)^{1/2}; \tag{9}$$

the suffix zero refers to parameters appropriate to transverse photons, while the suffix one refers to parameters appropriate to longitudinal plasmons; q, μ, and $\boldsymbol{v}$ are the charge, mass, and the velocity of the electron; N refers to the number of transverse photons or longitudinal plasmons of energy $\hbar\omega$, momentum $\hbar\boldsymbol{k}$, and polarization vector $\boldsymbol{\epsilon}$; D is the dielectric coefficient of the plasma appropriate to photons or plasmons; and L^3 is the volume of the plasma under study.

According to the Heitler–Ma formulation of the quantum time-dependent perturbation theory, the transition probability $j(f, i)$ from an initial state $|\text{i}\rangle$ of energy E_i to a final state $|\text{f}\rangle$ of energy E_f is given by (Heitler 1954):

$$j(\text{f}, \text{i}) = (2\pi/\hbar)|\langle \text{f}|T|\text{i}\rangle|^2 \delta(E_\text{f} - E_\text{i}), \tag{10}$$

where

$$\langle \text{f}|T|\text{i}\rangle = \langle \text{f}|H_{\text{int}}|\text{i}\rangle + \sum_{\text{f}'\neq \text{i}} \frac{\langle \text{f}|H_{\text{int}}|\text{f}'\rangle\langle \text{f}'|T|\text{i}\rangle}{E_\text{i} - E_{\text{f}'}}, \tag{11}$$

where $|\text{f}'\rangle$ is an intermediate state of energy $E_{\text{f}'}$. The series solution of the integral equation (11) may be written

$$\langle \text{f}|T|\text{i}\rangle = \langle \text{f}|H_{\text{int}}|\text{i}\rangle + \sum_{\text{f}'\neq \text{i}} \frac{\langle \text{f}|H_{\text{int}}|\text{f}'\rangle\langle \text{f}'|H_{\text{int}}|\text{i}\rangle}{E_\text{i} - E_{\text{f}'}} + \sum_{\text{f}'\neq \text{i}}\sum_{\text{f}''\neq \text{i}} \frac{\langle \text{f}|H_{\text{int}}|\text{f}'\rangle\langle \text{f}'|H_{\text{int}}|\text{f}''\rangle\langle \text{f}''|H_{\text{int}}|\text{i}\rangle}{(E_\text{i} - E_{\text{f}'})(E_\text{i} - E_{\text{f}''})} + \ldots, \tag{12}$$

where $|\text{f}''\rangle$ is another intermediate state of energy $E_{\text{f}''}$. In this paper we are interested only in the second term of the right-hand side of (12), and we shall see later that this term is the one that is responsible for mode conversion by double Čerenkov processes in a uniform, homogeneous and isotropic plasma.

3. Theory of mode conversion

It is apparent from (3) to (12) that the first term on the right-hand side of (12) (ie, $\langle \mathrm{f}|H_{\mathrm{int}}|\mathrm{i}\rangle$) can only lead to (a spontaneous plus stimulated) emission and absorption of a transverse photon or a longitudinal plasmon. This term clearly cannot lead to a mode conversion. That is, this term cannot lead to an absorption of a quantum of one kind (either the photon or the plasmon) and to a subsequent emission of a quantum of the other kind (either the plasmon or the photon, respectively). Let us then examine the nature of the second term on the right-hand side of (12). From (3) we get

$$\begin{aligned}\langle \mathrm{f}|H_{\mathrm{int}}|\mathrm{f}'\rangle\langle \mathrm{f}'|H_{\mathrm{int}}|\mathrm{i}\rangle &= \langle \mathrm{f}|H_0+H_1|\mathrm{f}'\rangle\langle \mathrm{f}'|H_0+H_1|\mathrm{i}\rangle \\ &= (\langle \mathrm{f}|H_0|\mathrm{f}'\rangle\langle \mathrm{f}'|H_0|\mathrm{i}\rangle+\langle \mathrm{f}|H_1|\mathrm{f}'\rangle\langle \mathrm{f}'|H_1|\mathrm{i}\rangle) \\ &\quad +(\langle \mathrm{f}|H_0|\mathrm{f}'\rangle\langle \mathrm{f}'|H_1|\mathrm{i}\rangle+\langle \mathrm{f}|H_1|\mathrm{f}'\rangle\langle \mathrm{f}'|H_0|\mathrm{i}\rangle). \end{aligned} \tag{13}$$

It is clear that the two terms in the first parenthesis on the right-hand side of (13) will lead to two-photon or two-plasmon processes, while the two terms in the second parenthesis on the right-hand side of (13) will lead to a mode conversion by double Čerenkov processes (ie, to the Čerenkov absorption of a photon or a plasmon and the subsequent Čerenkov emission of a plasmon or a photon, respectively). Thus the matrix element $\langle \mathrm{f}|T|\mathrm{i}\rangle$ appropriate to the mode conversion processes under study may be written

$$\langle \mathrm{f}|T|\mathrm{i}\rangle = \frac{\langle \mathrm{f}|H_0|\mathrm{f}'\rangle\langle \mathrm{f}'|H_1|\mathrm{i}\rangle}{E_{\mathrm{i}}-E_{\mathrm{f}'}}+\frac{\langle \mathrm{f}|H_1|\mathrm{f}''\rangle\langle \mathrm{f}''|H_0|\mathrm{i}\rangle}{E_{\mathrm{i}}-E_{\mathrm{f}''}}. \tag{14}$$

In order to treat this problem in the most general way, let us suppose that the initial state $|\mathrm{i}\rangle$ represents a state in which there are N_0 photons, N_1 plasmons, and an electron moving with the velocity $\boldsymbol{v}_{\mathrm{i}}$. That is, $|\mathrm{i}\rangle = |N_0, N_1, \boldsymbol{v}_{\mathrm{i}}\rangle$. For example, these initial N_0 photons may be due to a large amplitude electromagnetic wave incident on a plasma (such as the incident laser radiation in a laser fusion experiment or a large amplitude electromagnetic wave used to heat a plasma) or it could be the thermally excited transverse radiation field in the plasma; while the initial N_1 plasmons may be due to large amplitude plasma waves excited in the plasma (say for example by the familiar two-stream instability in beam–plasma systems or a large amplitude plasma wave launched with a probe in the plasma) or it could be the thermally excited plasmon field in the plasma.

Let us first consider the mode conversion process where a transverse electromagnetic wave (ie, a photon) is converted into a longitudinal plasma wave (ie, a plasmon) by the double Čerenkov processes. Then in the final state $|\mathrm{f}\rangle$ there will be (N_0-1) photons, (N_1+1) plasmons, and the electron moving with a velocity $\boldsymbol{v}_{\mathrm{f}}$. That is

$$|\mathrm{f}\rangle = |N_0-1, N_1+1, \boldsymbol{v}_{\mathrm{f}}\rangle.$$

It may be pointed out that the final velocity $\boldsymbol{v}_{\mathrm{f}}$ of the electron will in general be different from its initial velocity $\boldsymbol{v}_{\mathrm{i}}$ since both energy and momentum must be conserved in the transition from the initial state $|\mathrm{i}\rangle$ to the final state $|\mathrm{f}\rangle$. It is seen from (14) that the transition from the initial state $|N_0, N_1, \boldsymbol{v}_\mathrm{i}\rangle$ to the final state $|N_0-1, N_1+1, \boldsymbol{v}_\mathrm{f}\rangle$ can take place through either of the two intermediate states

$$|\mathrm{f}'\rangle = |N_0, N_1+1, \boldsymbol{v}'\rangle \tag{15}$$

and

$$|\mathrm{f}''\rangle = |N_0-1, N_1, \boldsymbol{v}''\rangle. \tag{16}$$

In reaching the final state $|\mathrm{f}\rangle$ from the initial state $|\mathrm{i}\rangle$ through the intermediate state $|\mathrm{f}'\rangle$ of (15) a plasmon is first emitted and a photon is subsequently absorbed, while in the passage through $|\mathrm{f}''\rangle$ of (16) a photon is first absorbed and a plasmon is subsequently emitted. On making use of (4) to (9), (15) and (16) in (14), we find (after somewhat lengthy and tedious algebra) that the matrix element appropriate to the conversion of a photon to a plasmon may be written

$$\begin{aligned}\langle N_0-1, N_1+1, \boldsymbol{v}_\mathrm{f}|T|N_0, N_1, \boldsymbol{v}_\mathrm{i}\rangle \\ = N_0^{1/2}(N_1+1)^{1/2}M_0M_1(\omega_1/k_1)\left(\frac{\boldsymbol{\epsilon}_0 \cdot \boldsymbol{v}_\mathrm{i}}{\hbar[\omega_0-\boldsymbol{k}_0 \cdot (\boldsymbol{v}_\mathrm{i}+\hbar\boldsymbol{k}_0/2\mu)]}\right. \\ \left. -\frac{\boldsymbol{\epsilon}_0 \cdot (\boldsymbol{v}_\mathrm{i}-\hbar\boldsymbol{k}_1/\mu)}{\hbar[\omega_1-\boldsymbol{k}_1 \cdot (\boldsymbol{v}_\mathrm{i}-\hbar\boldsymbol{k}_1/2\mu)]}\right)\delta_{\boldsymbol{v}_\mathrm{f}, \boldsymbol{v}_\mathrm{i}+\hbar(\boldsymbol{k}_0-\boldsymbol{k}_1)/\mu}.\end{aligned} \tag{17}$$

Hence from (17) and (10), the transition probability for the mode conversion of a photon to a plasmon may be written

$$\begin{aligned}j(N_0-1, N_1+1, \boldsymbol{v}_\mathrm{f}; N_0, N_1, \boldsymbol{v}_\mathrm{i}) \\ = (2\pi/\hbar)N_0(N_1+1)M_0^2M_1^2(\omega_1/k_1)^2\left|\frac{\boldsymbol{\epsilon}_0 \cdot \boldsymbol{v}_\mathrm{i}}{\hbar[\omega_0-\boldsymbol{k}_0 \cdot (\boldsymbol{v}_\mathrm{i}+\hbar\boldsymbol{k}_0/2\mu)]}\right. \\ \left. -\frac{\boldsymbol{\epsilon}_0 \cdot (\boldsymbol{v}_\mathrm{i}-\hbar\boldsymbol{k}_1/\mu)}{\hbar[\omega_1-\boldsymbol{k}_1 \cdot (\boldsymbol{v}_\mathrm{i}-\hbar\boldsymbol{k}_1/2\mu)]}\right|^2 \\ \times\delta\{\hbar(\omega_0-\omega_1)-\hbar(\boldsymbol{k}_0-\boldsymbol{k}_1) \cdot [\boldsymbol{v}_\mathrm{i}+\hbar(\boldsymbol{k}_0-\boldsymbol{k}_1)/2\mu]\}.\end{aligned} \tag{18}$$

We note that in (18)

$$\begin{aligned}\delta\{(\omega_0-\omega_1)-(\boldsymbol{k}_0-\boldsymbol{k}_1) \cdot [\boldsymbol{v}_\mathrm{i}+\hbar(\boldsymbol{k}_0-\boldsymbol{k}_1)/2\mu]\} \\ = \delta\{[\omega_0-\boldsymbol{k}_0 \cdot (\boldsymbol{v}_\mathrm{i}+\hbar\boldsymbol{k}_0/2\mu)]-[\omega_1-\boldsymbol{k}_1 \cdot (\boldsymbol{v}_\mathrm{i}-\hbar\boldsymbol{k}_1/2\mu)]+(\hbar\boldsymbol{k}_0 \cdot \boldsymbol{k}_1/\mu)\}.\end{aligned} \tag{19}$$

If we now make the reasonable assumption that $|\hbar\boldsymbol{k}_0 \cdot \boldsymbol{k}_1/\mu| \ll |\omega_0-\boldsymbol{k}_0 \cdot (\boldsymbol{v}_\mathrm{i}+\hbar\boldsymbol{k}_0/2\mu)|$, (18) becomes

$$\begin{aligned}j(N_0-1, N_1+1, \boldsymbol{v}_\mathrm{f}; N_0, N_1, \boldsymbol{v}_\mathrm{i}) \\ \simeq \frac{2\pi}{\hbar}N_0(N_1+1)M_0^2M_1^2\left(\frac{\omega_1}{\mu}\right)^2\left|\frac{\boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_1}{\omega_0-\boldsymbol{k}_0 \cdot (\boldsymbol{v}_\mathrm{i}+\hbar\boldsymbol{k}_0/2\mu)}\right|^2\frac{1}{\hbar}\delta\{(\omega_0-\omega_1) \\ -(\boldsymbol{k}_0-\boldsymbol{k}_1) \cdot [\boldsymbol{v}_\mathrm{i}+\hbar(\boldsymbol{k}_0-\boldsymbol{k}_1)/2\mu]\},\end{aligned} \tag{20}$$

where we have made use of the fact that the polarization vector of the longitudinal plasmon $\boldsymbol{\epsilon}_1 = \boldsymbol{k}_1/k_1$. It is interesting to note from (20) that the transition probability for the mode conversion of a photon to a plasmon is proportional to $|\boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_1|^2$. That is, the wavevector $\boldsymbol{k}_1$ of the newly born plasmon tends to be predominantly in the direction of the electric field $\boldsymbol{\epsilon}_0$ of the absorbed photon. Furthermore, it may be noted from (20) that the transition probability for the mode conversion of a photon to a plasmon is proportional to (N_1+1). The term proportional to N_1 shows that it is possible to have 'stimulated mode conversion'; that is, mode conversion of photons to plasmons is enhanced if there are plasmons present in the initial state.

Equation (20) is the transition probability for the mode conversion of a photon into a plasmon by an electron moving with a velocity $\boldsymbol{v}_\mathrm{i}$. We now wish to calculate the differential cross section for this mode conversion process. For this purpose we note that the incident flux of photons is $N_0 c/L^3$ and

$$\sum_{\boldsymbol{k}}(\ldots) \to \left(\frac{L}{2\pi}\right)^3 \int \mathrm{d}k k^2 \, \mathrm{d}\Omega_k(\ldots), \tag{21}$$

where c is the velocity of light and $\mathrm{d}\Omega_k$ is the element of solid angle. Hence the differential cross section for this mode conversion process may be written

$$\frac{\mathrm{d}\sigma(\boldsymbol{v}_\mathrm{i})}{\mathrm{d}\Omega_{k_1}} = \left(\frac{L^3}{N_0 c}\right)\left(\frac{L}{2\pi}\right)^3 \int \mathrm{d}k_1 \, k_1^2 j(N_0-1, N_1+1, \boldsymbol{v}_\mathrm{f}; N_0, N_1, \boldsymbol{v}_\mathrm{i}). \tag{22}$$

Let $F(\boldsymbol{v}_\mathrm{i})$ represent the probability that the electron is in the initial state $|\boldsymbol{v}_\mathrm{i}\rangle$. We will assume that the probability function $F(\boldsymbol{v}_\mathrm{i})$ is normalized such that

$$\int \mathrm{d}\boldsymbol{v}_\mathrm{i} \, F(\boldsymbol{v}_\mathrm{i}) = 1. \tag{23}$$

Thus from (22) and (23), the 'average differential cross section' for this mode conversion process may be written

$$\left\langle \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{k_1}} \right\rangle = \mathrm{P} \int \mathrm{d}\boldsymbol{v}_\mathrm{i} \, F(\boldsymbol{v}_\mathrm{i}) \frac{\mathrm{d}\sigma(\boldsymbol{v}_\mathrm{i})}{\mathrm{d}\Omega_{k_1}}, \tag{24}$$

where P denotes the principal value. The reason for taking the principal value in (24) is apparent from the restriction $\mathrm{f}' \neq \mathrm{i}$ in the sum over f' in the second term on the right-hand side of (12).

It is seen from (20) and (22) that the exact evaluation of the differential cross section $\mathrm{d}\sigma(\boldsymbol{v}_\mathrm{i})/\mathrm{d}\Omega_{k_1}$ is extremely difficult. Hence let us now make a rough estimate of this differential cross section. For this purpose we will make the reasonable assumption that the wavelength of the electromagnetic waves is very much larger than the wavelength of the plasma waves; that is we will assume that $k_0 \ll k_1$. Furthermore, we will assume that $v_\mathrm{i} \gg \hbar k_0/2\mu$ and $v_\mathrm{i} \gg \hbar k_1/2\mu$; that is, we will neglect the terms corresponding to Compton recoil. Then the approximate form of (20) becomes

$$\begin{aligned} &j(N_0-1, N_1+1, \boldsymbol{v}_\mathrm{f}; N_0, N_1, \boldsymbol{v}_\mathrm{i}) \\ &\qquad \simeq (2\pi/\hbar^2) N_0 (N_1+1) M_0^2 M_1^2 (\omega_1/\mu)^2 |\boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_1/\omega_0|^2 \delta(\omega_0 - \omega_1 + \boldsymbol{k}_1 \cdot \boldsymbol{v}_\mathrm{i}). \end{aligned} \tag{25}$$

The dielectric coefficient $D \simeq 1-\omega_\mathrm{p}^2/\omega^2$, where ω_p is the plasma frequency. Hence

$$\frac{\partial}{\partial\omega}(\omega^2 D) \simeq 2\omega. \tag{26}$$

Thus on making use of (8), (9), (25) and (26) in (22) we get

$$\frac{d\sigma(\boldsymbol{v}_i)}{d\Omega_{k_1}} \simeq \int dk_1 \frac{k_1^2(N_1+1)q^4\omega_1|\boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_1|^2}{c\mu^2\omega_0^3}\delta(\omega_0-\omega_1+\boldsymbol{k}_1 \cdot \boldsymbol{v}_i)$$
$$\simeq |\boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_1|^2(N_1+1)(q^2/\mu c^2)^2(c/|\boldsymbol{v}_i|)^3(\omega_1/\omega_0)|(\omega_0-\omega_1)/\omega_0|^2. \quad (27)$$

The restriction imposed by the Dirac δ function $\delta(\omega_0-\omega_1+\boldsymbol{k}_1 \cdot \boldsymbol{v}_i)$ implies that the result of (27) is valid if and only if $|\omega_0-\omega_1| \leqslant k_1 v_i$. Hence the average differential cross section for the mode conversion of a photon into a plasmon by the electrons in a plasma may be written

$$\left\langle \frac{d\sigma}{d\Omega_{k_1}} \right\rangle \simeq |\boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_1|^2(N_1+1)(q^2/\mu c^2)^2(c/v_T)^3(\omega_p/\omega_0)|(\omega_0-\omega_p)/\omega_0|^2 \quad (28)$$

for $|\omega_0-\omega_p| \leqslant k_D v_T = \omega_p$, where k_D is the Debye wavenumber and v_T is the average thermal velocity of the electrons in the plasma. Here we have assumed that the frequency of the plasmon ω_1 is approximately equal to the plasma frequency ω_p and the wavenumber of the plasmon $k_1 \leqslant k_D$ (ie, the plasmon wavelength must be greater than or equal to the plasma Debye length). Here again we may note from (28) that the polarization vector $\boldsymbol{\epsilon}_1$ of the newly born plasmon tends to be predominantly in the direction of the polarization vector $\boldsymbol{\epsilon}_0$ of the absorbed photon. It is seen from (28) that $\langle d\sigma/d\Omega_{k_1}\rangle$ is proportional to (N_1+1). The term proportional to one is due to Čerenkov absorption of the photon and the subsequent spontaneous Čerenkov emission of the plasmon, while the term proportional to N_1 is due to Čerenkov absorption of the photon and the subsequent stimulated Čerenkov emission of the plasmon. Thus, there is not only a spontaneous mode conversion but also a stimulated mode conversion. Since for most laboratory or fusion plasmas $v_T \ll c$, we see from (28) that the cross section for the mode conversion of a photon to a plasmon will in general be very much larger than the square of the classical electron radius $(q^2/\mu c^2)^2$ for $|\omega_0-\omega_p|$ comparable to ω_p.

Let us now consider the other mode conversion process where a longitudinal plasma wave (ie, a plasmon) is converted into a transverse electromagnetic wave (ie, a photon). This process may also be looked upon as the emission of transverse electromagnetic waves by longitudinal plasma oscillations. This may occur in a laboratory beam–plasma system, or in extraterrestrial plasmas (such as the observed intense radio wave emission from the bright areas of the sun). It is known that large amplitude electron plasma oscillations can be excited in non-equilibrium plasmas (for example by the familiar two-stream instability or the inverse Landau damping). By mode conversion the large amplitude plasma oscillations can be transformed to transverse electromagnetic waves, and thus a non-equilibrium plasma may emit various forms of non-thermal radiation.

Let us again start from the same initial state $|i\rangle = |N_0, N_1, \boldsymbol{v}_i\rangle$ of N_0 photons, N_1 plasmons, and an electron moving with a velocity $\boldsymbol{v}_i$. Throughout this paper we consistently use the suffix zero to denote parameters appropriate to photons and the suffix one to denote parameters appropriate to plasmons. The final state for the mode conversion of a plasmon to a photon is $|f\rangle = |N_0+1, N_1-1, \boldsymbol{v}_f\rangle$, that is a state of (N_0+1) photons, (N_1-1) plasmons and the electron moving with the velocity $\boldsymbol{v}_f$. This mode conversion process can occur through either of the two intermediate states $|f'\rangle = |N_0, N_1-1, \boldsymbol{v}'\rangle$ and $|f''\rangle = |N_0+1, N_1, \boldsymbol{v}''\rangle$. Thus one can show (after a somewhat lengthy and tedious algebra) that the matrix element appropriate to the conversion of a plasmon to a photon may be written

$$\langle N_0+1, N_1-1, \boldsymbol{v}_f|T|N_0, N_1, \boldsymbol{v}_i\rangle = (N_0+1)^{1/2}N_1^{1/2}M_0M_1(\omega_1/k_1)\left(\frac{\boldsymbol{\epsilon}_0\cdot(\boldsymbol{v}_i+\hbar\boldsymbol{k}_1/\mu)}{\hbar[\omega_1-\boldsymbol{k}_1\cdot(\boldsymbol{v}_i+\hbar\boldsymbol{k}_1/2\mu)]} - \frac{\boldsymbol{\epsilon}_0\cdot\boldsymbol{v}_i}{\hbar[\omega_0-\boldsymbol{k}_0\cdot(\boldsymbol{v}_i-\hbar\boldsymbol{k}_0/2\mu)]}\right)\delta_{\boldsymbol{v}_f,\boldsymbol{v}_i-\hbar(\boldsymbol{k}_0-\boldsymbol{k}_1)/\mu}. \tag{29}$$

The transition probability for the mode conversion of a plasmon to a photon may be written

$$j(N_0+1, N_1-1, \boldsymbol{v}_f; N_0, N_1, \boldsymbol{v}_i) = (2\pi/\hbar)|\langle N_0+1, N_1-1, \boldsymbol{v}_f|T|N_0, N_1, \boldsymbol{v}_i\rangle|^2 \times \delta\{\hbar(\omega_0-\omega_1)-\hbar(\boldsymbol{k}_0-\boldsymbol{k}_1)\cdot[\boldsymbol{v}_i-\hbar(\boldsymbol{k}_0-\boldsymbol{k}_1)/2\mu]\}. \tag{30}$$

Here again we will make the reasonable assumption that

$$|\hbar\boldsymbol{k}_0\cdot\boldsymbol{k}_1/\mu| \ll |\omega_0-\boldsymbol{k}_0\cdot(\boldsymbol{v}_i-\hbar\boldsymbol{k}_0/2\mu)|.$$

Then on making use of (29), (30) becomes

$$j(N_0+1, N_1-1, \boldsymbol{v}_f; N_0, N_1, \boldsymbol{v}_i) \simeq (2\pi/\hbar)N_1(N_0+1)M_0^2M_1^2(\omega_1/\mu)^2\left|\frac{\boldsymbol{\epsilon}_0\cdot\boldsymbol{\epsilon}_1}{\omega_0-\boldsymbol{k}_0\cdot(\boldsymbol{v}_i-\hbar\boldsymbol{k}_0/2\mu)}\right|^2\frac{1}{\hbar} \times \delta\{(\omega_0-\omega_1)-(\boldsymbol{k}_0-\boldsymbol{k}_1)\cdot[\boldsymbol{v}_i-\hbar(\boldsymbol{k}_0-\boldsymbol{k}_1)/2\mu]\}, \tag{31}$$

where $\boldsymbol{\epsilon}_1 = \boldsymbol{k}_1/k$. The incident flux of plasmons is N_1v_g/L^3, where $v_g = \partial\omega_1/\partial k_1$ is the group velocity of the plasmons. Thus on making use of (21) to sum over the final photon states, the differential cross section for the mode conversion of a plasmon to a photon may be written

$$\frac{d\sigma(\boldsymbol{v}_i)}{d\Omega_{k_0}} = \left(\frac{L^3}{N_1v_g}\right)\left(\frac{L}{2\pi}\right)^3\int dk_0k_0^2\,j(N_0+1, N_1-1, \boldsymbol{v}_f; N_0, N_1, \boldsymbol{v}_i) = \left(\frac{L^3}{N_1v_g}\right)\left(\frac{L}{2\pi c}\right)^3\int d\omega_0\omega_0^2\,j(N_0+1, N_1-1, \boldsymbol{v}_f; N_0, N_1, \boldsymbol{v}_i) \tag{32}$$

since for photons $\omega_0 = ck_0$. Thus the 'average differential cross section' for this mode conversion process may be written

$$\left\langle\frac{d\sigma}{d\Omega_{k_0}}\right\rangle = \mathrm{P}\int d\boldsymbol{v}_i\,F(\boldsymbol{v}_i)\frac{d\sigma(\boldsymbol{v}_i)}{d\Omega_{k_0}}, \tag{33}$$

where P denotes the principal value.

It is clear from (31) and (32) that the exact evaluation of $d\sigma(\boldsymbol{v}_i)/d\Omega_{k_0}$ is extremely difficult. Thus we will now make a rough estimate of this differential cross section. For this purpose we will neglect the terms corresponding to Compton recoil, that is we will assume that $v_i \gg \hbar k_0/2\mu$ and $v_i \gg \hbar k_1/2\mu$. Thus on making use of (8), (9), (26) and (31) in (32) we get

$$\frac{d\sigma(\boldsymbol{v}_i)}{d\Omega_{k_0}} \simeq \int d\omega_0 \left(\frac{\omega_0(N_0+1)q^4\omega_1|\boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_1|^2}{c^3 v_g \mu^2 |\omega_0(1-\boldsymbol{k}_0 \cdot \boldsymbol{v}_i/k_0 c)|^2} \delta[\omega_0(1-\boldsymbol{k}_0 \cdot \boldsymbol{v}_i/k_0 c) - (\omega_1 - \boldsymbol{k}_1 \cdot \boldsymbol{v}_i)] \right)$$

$$= |\boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_1|^2 (N_0+1)(q^2/\mu c^2)^2 (c/v_g)[\omega_1/|(\omega_1 - \boldsymbol{k}_1 \cdot \boldsymbol{v}_i)(1-\boldsymbol{k}_0 \cdot \boldsymbol{v}_i/k_0 c)|]. \quad (34)$$

The restriction imposed by the Dirac δ function $\delta[\omega_0(1-\boldsymbol{k}_0 \cdot \boldsymbol{v}_i/k_0 c)-(\omega_1 - \boldsymbol{k}_1 \cdot \boldsymbol{v}_i)]$ implies that the frequency ω_0 of the newly born photon is such that

$$\omega_0(1-\boldsymbol{k}_0 \cdot \boldsymbol{v}_i/k_0 c) = (\omega_1 - \boldsymbol{k}_1 \cdot \boldsymbol{v}_i).$$

Since for most cases of practical interest $\omega_1^2 \simeq \omega_p^2[1+3(k_1\lambda_D)^2] \gg (k_1 v_T)^2$ and $(v_T/c) \ll 1$, the average differential cross section for mode conversion of a plasmon into a photon by the electrons in a plasma may be written

$$\langle d\sigma/d\Omega_{k_0} \rangle \simeq |\boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_1|^2 (N_0+1)(q^2/\mu c^2)^2 (c/v_g), \quad (35)$$

where λ_D is the Debye length. The frequency ω_0 of the newly born photon is $\omega_0 \simeq \omega_p[1+\frac{3}{2}(k_1\lambda_D)^2]$. It is apparent from (35) that the polarization vector $\boldsymbol{\epsilon}_0$ of the newly born photon tends to be predominantly in the direction of the polarization vector $\boldsymbol{\epsilon}_1$ of the parent plasmon. Here again we see from (35) that $\langle d\sigma/d\Omega_{k_0} \rangle$ is proportional to (N_0+1). The term proportional to one represents the spontaneous mode conversion process, while the term proportional to N_0 represents the stimulated mode conversion process. Since $\omega_1^2 \simeq \omega_p^2[1+3(k_1\lambda_D)^2]$, the group velocity

$$v_g = \partial\omega_1/\partial k_1 \simeq \{(\omega_p\lambda_D)(3k_1\lambda_D)/[1+\tfrac{3}{2}(k_1\lambda_D)^2]\} = \{v_T(3k_1\lambda_D)/[1+\tfrac{3}{2}(k_1\lambda_D)^2]\}.$$

Thus we see from (35) that the cross section for the mode conversion of a plasmon to a photon will in general be very much larger than the square of the classical electron radius $(q^2/\mu c^2)^2$ since $v_T \ll c$ and $k_1\lambda_D \leqslant 1$ for most cases of practical interest.

4. Discussion

It is seen from (28) and (35) that the average differential cross sections for the mode conversions by double Čerenkov processes in plasmas can in general be very much larger than the square of the classical electron radius $(q^2/\mu c^2)^2$. We now wish to show that for most cases of practical interest (ie, for laboratory plasmas, for laser fusion plasmas and for plasma parameters in the solar corona) stimulated mode conversion will in general exceed spontaneous mode conversion. For this purpose we now consider a plasma at thermodynamic equilibrium. Then one can show (Bekefi 1966) that in the classical limit (ie, $\hbar\omega \ll \kappa T$) the energy density U per unit frequency interval is

$$U_0 = V(\kappa T\omega^2/2\pi^2c^3)(1-\omega_p^2/\omega^2)^{1/2} \quad (36)$$

for photons and

$$U_1 = V(\kappa T\omega^2/2\pi^2c^3)3^{-3/2}(c/v_T)^3(1-\omega_p^2/\omega^2)^{1/2} \quad (37)$$

for plasmons of frequency $\omega \sim \omega_p$. Here V is the volume of the plasma. However, in non-equilibrium plasmas the energy densities will in general be very much larger than those given by (36) and (37). Thus the results of (36) and (37) represent the lower bound on energy densities that can be found in plasmas. It may be noted that for frequencies ω

near the plasma frequency ω_p the energy density in the longitudinal mode U_1 is approximately $(c/v_T)^3$ greater than that in the transverse mode U_0. Thus for $\omega \sim \omega_p$, in general the stimulated mode conversion of photons into plasmons will be much more efficient than the reverse conversion of plasmons into photons. Since in general $k_1 \gg k_0$, only the number N of quanta (either the photons or the plasmons) in the frequency interval $\Delta\omega \simeq \hbar k_1^2/2\mu$ where $k_1 \simeq (\omega_0 - \omega_p)/v_T$ can take part in the stimulated mode conversion processes. That is, the frequency interval of interest

$$\Delta\omega \simeq (\hbar/2\kappa T)(\omega_0 - \omega_p)^2. \tag{38}$$

Since the number of quanta $N = U\Delta\omega/\hbar\omega$, it is relatively easy to show from (36), (37) and (38) that for equilibrium plasma N_1 of (28) and N_0 of (35) are given by

$$N_1 \simeq (V/12\pi^2)(\omega_0 - \omega_p)^3/v_T^3 \tag{39}$$

and

$$N_0 \simeq (V3^{1/2}/4\pi^2)(\omega_0 - \omega_p)^3\omega_0/c^3\omega_p, \tag{40}$$

respectively. Here V is the volume of interaction between the incident flux of quanta (either the photons or the plasmons) and the plasma.

Let us now consider the mode conversion of photons into plasmons in a laboratory plasma of electron density $n_0 \simeq 10^{12}\,\mathrm{cm}^{-3}$ and an electron temperature $T \simeq 4\,\mathrm{eV}$. Further let us assume that the incident beam of electromagnetic energy has a cross sectional area of $10\,\mathrm{cm}^2$ and the frequency of the incident photons $\omega_0 \simeq 1{\cdot}1\omega_p$. If the incident electromagnetic beam traverses a distance l in the plasma, then the fraction f of the incident electromagnetic wave energy that will be converted into electrostatic plasma waves is given by

$$f \simeq n_0\langle \mathrm{d}\sigma/\mathrm{d}\Omega_{k_1}\rangle l. \tag{41}$$

For these conditions we find that $\langle \mathrm{d}\sigma/\mathrm{d}\Omega_{k_1}\rangle \simeq (N_1+1)\times 9{\cdot}52\times 10^{-21}\,\mathrm{cm}^2$, $N_1 \simeq 2{\cdot}55\times 10^3 V = 2{\cdot}55\times 10^4 l$, and $f \simeq 9{\cdot}52\times 10^{-9} l(2{\cdot}55\times 10^4 l + 1)$. The distance l in which the entire incident electromagnetic wave energy is converted into electrostatic electron plasma waves (ie, the distance l for which $f \simeq 1$) is $l \simeq 82{\cdot}2$ cm. For reasonable values of l since $N_1 \gg 1$, we see that stimulated mode conversion is considerably larger than spontaneous mode conversion. The fact that the entire incident flux of photons is converted into plasmons in a distance of about 82·2 cm seems to indicate that this nonlinear mode conversion by double Čerenkov processes in plasmas is a highly efficient process. By Landau damping these converted electrostatic plasma waves will then heat the plasma. Thus it appears that one should be able to heat a plasma via this nonlinear mode conversion process by shining on the plasma electromagnetic waves of frequency ω_0 slightly above the plasma frequency ω_p.

Let us now consider the mode conversion of photons into plasmons in typical laser fusion experiments. In present day experiments on laser produced plasmas one obtains an electron density $n_0 \simeq 10^{19}\,\mathrm{cm}^{-3}$ and an electron temperature $T \simeq 100\,\mathrm{eV}$. Let us assume that the incident laser beam is focused on a cross sectional area of $10^{-4}\,\mathrm{cm}^2$ and the frequency of the incident photons $\omega_0 \simeq 1{\cdot}1\,\omega_p$. For these conditions we find that $\langle \mathrm{d}\sigma/\mathrm{d}\Omega_{k_1}\rangle \simeq (N_1+1)\times 7{\cdot}66\times 10^{-23}\,\mathrm{cm}^2$, $N_1 \simeq 6{\cdot}59\times 10^{11}\, V = 6{\cdot}59\times 10^7 l$, and $f \simeq 7{\cdot}66\times 10^{-4} l(6{\cdot}59\times 10^7 l + 1)$. Hence the distance l in which the entire incident electromagnetic wave energy is converted into electrostatic electron plasma waves (ie, the distance l for which $f \simeq 1$) is $l \simeq 4{\cdot}44\times 10^{-3}$ cm. Here again we see that for this value of l, $N_1 \gg 1$, and consequently stimulated mode conversion is considerably larger than spontaneous mode conversion. By Landau damping these converted electrostatic plasma waves will then heat the laser produced plasma.

Let us now consider the mode conversion of plasmons into photons in the solar corona. The generally accepted view is that an eruption from the chromosphere or lower corona yields a beam of electrons in the range of 20 to 200 keV. This electron beam travelling upward into the corona can excite large amplitude electron plasma oscillations by the familiar two-stream instability. According to (35) these plasma oscillations can undergo mode conversion into transverse electromagnetic waves of frequency $\omega_0 \sim \omega_p$. In the solar corona the electron density $n_0 \simeq 4 \times 10^9\ \mathrm{cm}^{-3}$ and the electron temperature $T \simeq 10^5\ \mathrm{K} \simeq 8{\cdot}6\ \mathrm{eV}$. Hence $\omega_p \simeq 3{\cdot}58 \times 10^9\ \mathrm{s}^{-1}$, $v_T \simeq 1{\cdot}74 \times 10^8\ \mathrm{cm\,s}^{-1}$, and the Debye length $\lambda_D \simeq 3{\cdot}45 \times 10^{-2}$ cm. If we assume a 100 keV electron beam, then the beam velocity $v_B \simeq 1{\cdot}9 \times 10^{10}\ \mathrm{cm\,s}^{-1}$. The wavenumber k_1 of the plasma oscillations that are excited by this electron beam is $k_1 \simeq \omega_p/v_B \simeq 1{\cdot}88 \times 10^{-1}\ \mathrm{cm}^{-1}$. Thus from (35) and (40) we obtain $\langle \mathrm{d}\sigma/\mathrm{d}\Omega_{k_0} \rangle \simeq (N_0+1) \times 7{\cdot}02 \times 10^{-22}\ \mathrm{cm}^2$,

$$N_0 \simeq 2{\cdot}03 \times 10^{-8}\ V \simeq 2{\cdot}03 \times 10^{-8\,3},$$

and $f \simeq 2{\cdot}81 \times 10^{-11} l(2{\cdot}03 \times 10^{-8} l^3 + 1)$, where l is the characteristic scale length over which large amplitude plasma oscillations are excited in the solar corona. The characteristic length scale l in which the entire electrostatic electron plasma wave energy will be converted into transverse electromagnetic waves (ie, the length scale l for which $f \simeq 1$) is $l \simeq 3{\cdot}64 \times 10^4$ cm. Here again one may note that for such scale lengths the stimulated mode conversion is considerably larger than the spontaneous mode conversion.

We remarked earlier that other types of mode conversions can occur in a plasma due to either a density discontinuity or a density gradient. According to Tidman (1960) the efficiency of mode conversion due to a density discontinuity is much larger than that due to a smooth density gradient. However according to Bekefi (1966): 'The efficiency with which the two waves couple at a boundary—that is the fraction of the energy transferred from one mode to the other—depends very sensitively on just about every plasma parameter. Therefore, there is hardly a theoretical model (and there are several in the literature) that can be readily applied to a given experimental situation'. For this reason we find it extremely difficult to compare mode conversion efficiencies of a volume process (such as the one presented in this paper) with those associated with surface effects (such as the one associated with a density discontinuity). However, according to Kritz and Mintzer (1960), for the conversion of longitudinal plasma waves into transverse electromagnetic waves at a density discontinuity in the solar corona, the plasma waves must be incident within zero degrees eighteen minutes with respect to the normal to the density discontinuity (ie, a rather stringent condition). In particular for incident angles larger than zero degrees eighteen minutes there is no conversion of plasmons into photons at the density discontinuity. Taking the ratio of the densities across the discontinuity to be 2:1, and for $\omega \simeq 1{\cdot}1\omega_p$ (where ω_p is the plasma frequency of the high dense medium) they find that $10^{-7} < f < 10^{-4}$ for incident angle θ in the range $0^\circ\ 5' < \theta < 0^\circ\ 18'$. Furthermore for $\theta = 0$ (ie, for a normally incident longitudinal wave) there is no mode conversion at all. But for conditions in the solar corona we showed earlier that for volume mode conversion by double Čerenkov processes $f \simeq 1$ for characteristic length scales $l \sim 10^4$ cm. It does therefore appear that the mode conversion efficiencies at a density discontinuity (a surface process) is smaller in magnitude than those due to double Čerenkov processes (a volume process with a large conversion cross section).

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